

Anti Lie-Trotter formula

Koenraad M.R. Audenaert^{1,2,*} and Fumio Hiai^{3,†}

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¹ Department of Mathematics, Royal Holloway University of London,
Egham TW20 0EX, United Kingdom

² Department of Physics and Astronomy, Ghent University,
S9, Krijgslaan 281, B-9000 Ghent, Belgium

³ Tohoku University (Emeritus),
Hakusan 3-8-16-303, Abiko 270-1154, Japan

Abstract

Let A and B be positive semidefinite matrices. The limit of the expression $Z_p := (A^{p/2} B^p A^{p/2})^{1/p}$ as p tends to 0 is given by the well known Lie-Trotter-Kato formula. A similar formula holds for the limit of $G_p := (A^p \# B^p)^{2/p}$ as p tends to 0, where $X \# Y$ is the geometric mean of X and Y . In this paper we study the complementary limit of Z_p and G_p as p tends to ∞ , with the ultimate goal of finding an explicit formula, which we call the anti Lie-Trotter formula. We show that the limit of Z_p exists and find an explicit formula in a special case. The limit of G_p is shown for 2×2 matrices only.

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*E-mail: koenraad.audenaert@rhul.ac.uk

†E-mail: hiai.fumio@gmail.com

1 Introduction

When H, K are lower bounded self-adjoint operators on a Hilbert space \mathcal{H} and H_+, K_+ are their positive parts, the sum of H and K can be given a precise meaning as a lower bounded self-adjoint operator on the subspace \mathcal{H}_0 , which is defined as the closure of $\text{dom } H_+^{1/2} \cap \text{dom } K_+^{1/2}$. We denote this formal sum as $H \dot{+} K$. Then the well-known Lie-Trotter-Kato product formula, as originally established in [18, 11] and refined by many authors, expresses the convergence

$$\lim_{n \rightarrow \infty} (e^{-tH/n} e^{-tK/n})^n = e^{-t(H \dot{+} K)} P_0, \quad t > 0,$$

in the strong operator topology (uniformly in $t \in [a, b]$ for any $0 < a < b$), where P_0 is the orthogonal projection onto \mathcal{H}_0 . Although this formula is usually stated for densely-defined H, K , the proof in [11] applies to the improper case (i.e., H, K are not densely-defined) as well, under the convention that $e^{-tH} = 0$ on $(\text{dom } H)^\perp$ for $t > 0$, and similarly for e^{-tK} .

The Lie-Trotter-Kato formula can easily be modified to symmetric form and with a continuous parameter as [8, Theorem 3.6]

$$\lim_{p \searrow 0} (e^{-ptH/2} e^{-ptK} e^{-ptH/2})^{1/p} = e^{-t(H \dot{+} K)} P_0, \quad t > 0.$$

When restricted to matrices (and to $t = 1$) this can be rephrased as

$$\lim_{p \searrow 0} (A^{p/2} B^p A^{p/2})^{1/p} = P_0 \exp(\log A \dot{+} \log B), \quad (1.1)$$

where A and B are positive semidefinite matrices (written as $A, B \geq 0$ below), P_0 is now the orthogonal projection onto the intersection of the supports of A, B and $\log A \dot{+} \log B$ is defined as $P_0(\log A)P_0 + P_0(\log B)P_0$.

When σ is an operator mean [13] corresponding to an operator monotone function f on $(0, \infty)$ such that $\alpha := f'(1)$ is in $(0, 1)$, the operator mean version of the Lie-Trotter-Kato product formula is the convergence [8, Theorem 4.11]

$$\lim_{p \searrow 0} (e^{-ptH} \sigma e^{-ptK})^{1/p} = e^{-t((1-\alpha)H \dot{+} \alpha K)}, \quad t > 0,$$

in the strong operator topology, for a bounded self-adjoint operator H and a lower-bounded self-adjoint operator K on \mathcal{H} . Although it is not known whether the above formula holds even when both H, K are lower bounded (and unbounded), we can verify that (1.1) has the operator mean version

$$\lim_{p \searrow 0} (A^p \sigma B^p)^{1/p} = P_0 \exp((1-\alpha) \log A \dot{+} \alpha \log B), \quad (1.2)$$

for matrices $A, B \geq 0$. A proof of (1.2) is supplied in an appendix of this paper since it is not our main theme.

In particular, let σ be the geometric mean $A \# B$ (introduced first in [17] and further discussed in [13]), corresponding to the operator monotone function $f(x) = x^{1/2}$ (hence $\alpha = 1/2$). Then (1.2) yields

$$\lim_{p \searrow 0} (A^p \# B^p)^{2/p} = P_0 \exp(\log A \dot{+} \log B), \quad (1.3)$$

which has the same right-hand side as (1.1).

It turns out that the convergence of both (1.1) and (1.3) is monotone in the log-majorization order. For $d \times d$ matrices $X, Y \geq 0$, the log-majorization relation $X \prec_{(\log)} Y$ means that

$$\prod_{i=1}^k \lambda_i(X) \leq \prod_{i=1}^k \lambda_i(Y), \quad 1 \leq k \leq d,$$

with equality for $k = d$, where $\lambda_1(X) \geq \dots \geq \lambda_d(X)$ are the eigenvalues of X sorted in decreasing order and counting multiplicities. The Araki-Lieb-Thirring inequality can be written in terms of log-majorization as

$$(A^{p/2} B^p A^{p/2})^{1/p} \prec_{(\log)} (A^{q/2} B^q A^{q/2})^{1/q} \quad \text{if } 0 < p < q, \quad (1.4)$$

for matrices $A, B \geq 0$, see [14, 3, 2]. One can also consider the complementary version of (1.4) in terms of the geometric mean. Indeed, for $A, B \geq 0$ we have [2]

$$(A^q \# B^q)^{2/q} \prec_{(\log)} (A^p \# B^p)^{2/p} \quad \text{if } 0 < p < q. \quad (1.5)$$

Hence, for matrices $A, B \geq 0$, we see that $Z_p := (A^{p/2} B^p A^{p/2})^{1/p}$ and $G_p := (A^p \# B^p)^{2/p}$ both tend to $P_0 \exp(\log A \dot{+} \log B)$ as $p \searrow 0$, with the former decreasing (by (1.4)) and the latter increasing (by (1.5)) in the log-majorization order.

The main topic of this paper is the complementary question about what happens to the limits of Z_p and G_p as p tends to ∞ instead of 0. Although this seems a natural mathematical problem, we have not been able to find an explicit statement of concern in the literature. It is obvious that if A and B are commuting then $G_p = AB = Z_p$, independently of $p > 0$. However, if A and B are not commuting, then the limit behavior of Z_p and its eigenvalues as $p \rightarrow \infty$ is of a rather complicated combinatorial nature, and that of G_p seems even more complicated.

The problem of finding an explicit formula, which we henceforth call the anti Lie-Trotter formula, also emerges from recent developments of new Rényi relative entropies relevant to quantum information theory. Indeed, the recent paper [4] proposed to generalize the Rényi relative entropy as

$$D_{\alpha,z}(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left(\rho^{\alpha/2z} \sigma^{(1-\alpha)/z} \rho^{\alpha/2z} \right)^z$$

for density matrices ρ, σ with two real parameters α, z , and discussed the limit formulas when α, z converge to some special values. The limit case of $D_{\alpha,z}(\rho \parallel \sigma)$ as $z \rightarrow 0$ with α fixed is exactly related to our anti Lie-Trotter problem.

The rest of the paper is organized as follows. In Section 2 we prove the existence of the limit of Z_p as $p \rightarrow \infty$ when A, B are $d \times d$ positive semidefinite matrices. In Section 3 we analyze the case when the limit eigenvalue list of Z_p becomes $\lambda_i(A)\lambda_i(B)$ ($1 \leq i \leq d$), the maximal case in the log-majorization order. In Section 4 we extend the existence of the limit of Z_p to that of $(A_1^{p/2} \cdots A_{m-1}^{p/2} A_m^p A_{m-1}^{p/2} \cdots A_1^{p/2})^{1/p}$ with more than two matrices. Finally in Section 5 we treat G_p ; however we can prove the existence of the limit of G_p as $p \rightarrow \infty$ only when A, B are 2×2 matrices, and the general case must be left unsettled. The paper contains two appendices. The first is a proof of a technical lemma stated in Section 2, and the second supplies the detailed proof of (1.2).

2 Limit of $(A^{p/2}B^pA^{p/2})^{1/p}$ as $p \rightarrow \infty$

Let A and B be $d \times d$ positive semidefinite matrices having the eigenvalues $a_1 \geq \cdots \geq a_d (\geq 0)$ and $b_1 \geq \cdots \geq b_d (\geq 0)$, respectively, sorted in decreasing order and counting multiplicities. Let $\{v_1, \dots, v_d\}$ be an orthonormal set of eigenvectors of A such that $Av_i = a_i v_i$ for $i = 1, \dots, d$, and $\{w_1, \dots, w_d\}$ an orthonormal set of eigenvectors of B in a similar way. Then A and B are diagonalized as

$$A = V \operatorname{diag}(a_1, \dots, a_d) V^* = \sum_{i=1}^d a_i v_i v_i^*, \quad (2.1)$$

$$B = W \operatorname{diag}(b_1, \dots, b_d) W^* = \sum_{i=1}^d b_i w_i w_i^*. \quad (2.2)$$

For each $p > 0$ define a positive semidefinite matrix

$$Z_p := (A^{p/2} B^p A^{p/2})^{1/p}, \quad (2.3)$$

whose eigenvalues are denoted as $\lambda_1(p) \geq \lambda_2(p) \geq \cdots \geq \lambda_d(p)$, again in decreasing order and counting multiplicities.

Lemma 2.1. *For every $i = 1, \dots, d$ the limit*

$$\lambda_i := \lim_{p \rightarrow \infty} \lambda_i(p) \quad (2.4)$$

exists, and $a_1 b_1 \geq \lambda_1 \cdots \geq \lambda_d \geq a_d b_d$.

Proof. Since $(a_1 b_1)^p I \geq A^{p/2} B^p A^{p/2} \geq (a_d b_d)^p I$, we have $a_1 b_1 \geq \lambda_i(p) \geq a_d b_d$ for all $i = 1, \dots, d$ and all $p > 0$. By the Araki-Lieb-Thirring inequality [3] (or the log-majorization [2]), for every $k = 1, \dots, d$ we have

$$\prod_{i=1}^k \lambda_i(p) \leq \prod_{i=1}^k \lambda_i(q) \quad \text{if } 0 < p < q. \quad (2.5)$$

Therefore, the limit η_k of $\prod_{i=1}^k \lambda_i(p)$ as $p \rightarrow \infty$ exists for any $k = 1, \dots, d$ so that $\eta_1 \geq \dots \geq \eta_d \geq 0$. Let m ($0 \leq m \leq d$) be the largest k such that $\eta_k > 0$ (with $m := 0$ if $\eta_1 = 0$). When $1 \leq k \leq m$, we have $\lambda_k(p) \rightarrow \eta_k/\eta_{k-1}$ (where $\eta_0 := 1$) as $p \rightarrow \infty$. When $m < d$, $\lambda_{m+1}(p) \rightarrow \eta_{m+1}/\eta_m = 0$ as $p \rightarrow \infty$. Hence $\lambda_k(p) \rightarrow 0$ for all $k > m$. Therefore, the limit of $\lambda_i(p)$ as $p \rightarrow \infty$ exists for any $i = 1, \dots, d$. The latter assertion is clear now. \square

Lemma 2.2. *The first eigenvalue in (2.4) is given by*

$$\lambda_1 = \max\{a_i b_j : (V^*W)_{ij} \neq 0\},$$

where $(V^*W)_{ij}$ denotes the (i, j) entry of V^*W .

Proof. Write $V^*W = [u_{ij}]$. We observe that

$$(V^*A^{p/2}B^pA^{p/2}V)_{ij} = \sum_{k=1}^d u_{ik} \bar{u}_{jk} a_i^{p/2} a_j^{p/2} b_k^p.$$

In particular,

$$(V^*A^{p/2}B^pA^{p/2}V)_{ii} = \sum_{k=1}^d |u_{ik}|^2 a_i^p b_k^p$$

and hence we have

$$\lambda_1(p)^p \leq \text{Tr } A^{p/2}B^pA^{p/2} = \sum_{i=1}^d \sum_{k=1}^d |u_{ik}|^2 a_i^p b_k^p \leq d^2 \max\{a_i^p b_k^p : u_{ik} \neq 0\},$$

where Tr is the usual trace functional on $d \times d$ matrices. Therefore,

$$\lambda_1(p) \leq d^{2/p} \max\{a_i b_k : u_{ik} \neq 0\}. \quad (2.6)$$

On the other hand, we have

$$d\lambda_1(p)^p \geq \text{Tr } A^{p/2}B^pA^{p/2} \geq \min\{|u_{ik}|^2 : u_{ik} \neq 0\} \max\{a_i^p b_k^p : u_{ik} \neq 0\}$$

so that

$$\lambda_1(p) \geq \left(\frac{\min\{|u_{ik}|^2 : u_{ik} \neq 0\}}{d} \right)^{1/p} \max\{a_i b_k : u_{ik} \neq 0\}. \quad (2.7)$$

Estimates (2.6) and (2.7) give the desired expression immediately. In fact, they prove the existence of the limit in (2.4) as well apart from Lemma 2.1. \square

In what follows, for each $k = 1, \dots, d$ we write $\mathcal{I}_d(k)$ for the set of all subsets I of $\{1, \dots, d\}$ with $|I| = k$. For $I, J \in \mathcal{I}_d(k)$ we denote by $(V^*W)_{I,J}$ the $k \times k$ submatrix of V^*W corresponding to rows in I and columns in J ; hence $\det(V^*W)_{I,J}$ denotes the corresponding minor of V^*W . We also write $a_I := \prod_{i \in I} a_i$ and $b_I := \prod_{i \in I} b_i$. Since $\det(V^*W) \neq 0$, note that for any $k = 1, \dots, d$ and any $I \in \mathcal{I}_d(k)$ we have $\det(V^*W)_{I,J} \neq 0$ for some $J \in \mathcal{I}_d(k)$, and that for any $J \in \mathcal{I}_d(k)$ we have $\det(V^*W)_{I,J} \neq 0$ for some $I \in \mathcal{I}_d(k)$.

Lemma 2.3. *For every $k = 1, \dots, d$,*

$$\lambda_1 \lambda_2 \cdots \lambda_k = \max\{a_I b_J : I, J \in \mathcal{I}_d(k), \det(V^*W)_{I,J} \neq 0\}. \quad (2.8)$$

Proof. For each $k = 1, \dots, d$ the antisymmetric tensor powers $A^{\wedge k}$ and $B^{\wedge k}$ (see [5]) are given in the form of diagonalizations as

$$A^{\wedge k} = V^{\wedge k} \text{diag}(a_I)_{I \in \mathcal{I}_d(k)} V^{\wedge k}, \quad B^{\wedge k} = W^{\wedge k} \text{diag}(b_I)_{I \in \mathcal{I}_d(k)} W^{\wedge k},$$

and the corresponding representation of the $\binom{n}{k} \times \binom{n}{k}$ unitary matrix $V^{*\wedge k} W^{\wedge k}$ is given by

$$(V^{*\wedge k} W^{\wedge k})_{I,J} = \det(V^*W)_{I,J}, \quad I, J \in \mathcal{I}_d(k).$$

Note that the largest eigenvalue of

$$((A^{\wedge k})^{p/2} (B^{\wedge k})^p (A^{\wedge k})^{p/2})^{1/p} = ((A^{p/2} B^p A^{p/2})^{1/p})^{\wedge k}$$

is $\lambda_1(p) \lambda_2(p) \cdots \lambda_k(p)$, whose limit as $p \rightarrow \infty$ is $\lambda_1 \lambda_2 \cdots \lambda_k$ by Lemma 2.1. Apply Lemma 2.2 to $A^{\wedge k}$ and $B^{\wedge k}$ to obtain expression (2.8). \square

Let \mathcal{H} be a d -dimensional Hilbert space (say, \mathbb{C}^d), k be an integer with $1 \leq k \leq d$, and $\mathcal{H}^{\wedge k}$ be the k -fold antisymmetric tensor of \mathcal{H} . We write $x_1 \wedge \cdots \wedge x_k$ ($\in \mathcal{H}^{\wedge k}$) for the antisymmetric tensor of $x_1, \dots, x_k \in \mathcal{H}$ (see [5]). The next lemma says that the Grassmannian manifold $G(k, d)$ is realized in the projective space of $\mathcal{H}^{\wedge k}$. Although the lemma might be known to specialists, we cannot find a precise explanation in the literature. So, for the convenience of the reader, we will present its sketchy proof in Appendix A based on [7].

Lemma 2.4. *There are constants $\alpha, \beta > 0$ (depending on only d and k) such that*

$$\alpha \|P - Q\| \leq \inf_{\theta \in \mathbb{R}} \|u_1 \wedge \cdots \wedge u_k - e^{\sqrt{-1}\theta} v_1 \wedge \cdots \wedge v_k\| \leq \beta \|P - Q\|$$

for all orthonormal sets $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ and the respective orthogonal projections P and Q onto $\text{span}\{u_1, \dots, u_k\}$ and $\text{span}\{v_1, \dots, v_k\}$, where $\|P - Q\|$ is the operator norm of $P - Q$ and $\|\cdot\|$ inside infimum is the norm on $\mathcal{H}^{\wedge k}$.

The main result of the paper is the next theorem showing the existence of limit for the anti version of (1.1).

Theorem 2.5. *For every $d \times d$ positive semidefinite matrices A and B the matrix Z_p in (2.3) converges as $p \rightarrow \infty$ to a positive semidefinite matrix.*

Proof. By replacing A and B with VAV^* and VBV^* , respectively, we may assume that $V = I$ and so

$$A = \text{diag}(a_1, \dots, a_d), \quad B = W \text{diag}(b_1, \dots, b_d) W^*.$$

Choose an orthonormal basis $\{u_1(p), \dots, u_d(p)\}$ of \mathbb{C}^d for which we have $Z_p u_i(p) = \lambda_i(p) u_i(p)$ for $1 \leq i \leq d$. Let λ_i be given in Lemma 2.1, and assume that $1 \leq k < d$ and $\lambda_1 \geq \dots \geq \lambda_k > \lambda_{k+1}$. Moreover, let $\lambda_1(Z_p^{\wedge k}) \geq \lambda_2(Z_p^{\wedge k}) \geq \dots$ be the eigenvalues of $Z_p^{\wedge k}$ in decreasing order. We note that

$$\begin{aligned} \lim_{p \rightarrow \infty} \lambda_1(Z_p^{\wedge k}) &= \lim_{p \rightarrow \infty} \lambda_1(p) \cdots \lambda_{k-1}(p) \lambda_k(p) \\ &= \lambda_1 \cdots \lambda_{k-1} \lambda_k \\ &> \lambda_1 \cdots \lambda_{k-1} \lambda_{k+1} = \lim_{p \rightarrow \infty} \lambda_2(Z_p^{\wedge k}). \end{aligned} \quad (2.9)$$

Hence it follows that $\lambda_1(Z_p^{\wedge k})$ is a simple eigenvalue of $Z_p^{\wedge k}$ for every p sufficiently large. Letting $w_{I,J} := \det W_{I,J}$ for $I, J \in \mathcal{I}_d(k)$ we compute

$$\begin{aligned} (Z_p^{\wedge k})^p &= (A^{\wedge k})^{p/2} W^{\wedge k} ((\text{diag}(b_1, \dots, b_d))^{\wedge k})^p (W^{\wedge k})^* (A^{\wedge k})^{p/2} \\ &= \text{diag}(a_I^{p/2})_I [w_{I,J}]_{I,J} \text{diag}(b_I^p)_I [\overline{w}_{J,I}]_{I,J} \text{diag}(a_I^{p/2})_I \\ &= \left[\sum_{K \in \mathcal{I}_d(k)} w_{I,K} \overline{w}_{J,K} a_I^{p/2} a_J^{p/2} b_K^p \right]_{I,J} \\ &= \eta_k^p \left[\sum_{K \in \mathcal{I}_d(k)} w_{I,K} \overline{w}_{J,K} \left(\frac{a_I^{1/2} a_J^{1/2} b_K}{\eta_k} \right)^p \right]_{I,J}, \end{aligned}$$

where $\eta_k := \lambda_1 \lambda_2 \cdots \lambda_k > 0$ so that

$$\eta_k = \max\{a_I b_K : I, K \in \mathcal{I}_d(k), w_{I,K} \neq 0\}$$

due to Lemma 2.3. We now define

$$\Delta_k := \{(I, K) \in \mathcal{I}_d(k)^2 : w_{I,K} \neq 0 \text{ and } a_I b_K = \eta_k\}.$$

Then we have

$$\begin{aligned} \left(\frac{Z_p^{\wedge k}}{\eta_k} \right)^p &= \left[\sum_{K \in \mathcal{I}_d(k)} w_{I,K} \overline{w}_{J,K} \left(\frac{a_I^{1/2} a_J^{1/2} b_K}{\eta_k} \right)^p \right]_{I,J} \\ &\longrightarrow Q := \left[\sum_{K \in \mathcal{I}_d(k)} w_{I,K} \overline{w}_{J,K} \delta_{I,J,K} \right]_{I,J}, \end{aligned}$$

where

$$\delta_{I,J,K} := \begin{cases} 1 & \text{if } (I, K), (J, K) \in \Delta_k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $Q_{I,I} \geq |w_{I,K}|^2 > 0$ when $(I, K) \in \Delta_k$, note that $Q \neq 0$. Furthermore, since the eigenvalue $\lambda_1(Z_p^{\wedge k})$ is simple (if p large), it follows from (2.9) that the limit Q of

$(Z_p^{\wedge k}/\eta_k)^p$ must be a rank one projection $\psi\psi^*$ up to a positive scalar multiple, where ψ is a unit vector in $(\mathbb{C}^d)^{\wedge k}$. Since the unit eigenvector $u_1(p) \wedge \cdots \wedge u_k(p)$ of $Z_p^{\wedge k}$ corresponding to the largest (simple) eigenvalue coincides with that of $(Z_p^{\wedge k}/\eta_k)^p$, we conclude that $u_1(p) \wedge \cdots \wedge u_k(p)$ converges ψ up to a scalar multiple $e^{\sqrt{-1}\theta}$. Therefore, by Lemma 2.4 the orthogonal projection onto $\text{span}\{u_1(p), \dots, u_k(p)\}$ converges as $p \rightarrow \infty$.

Assume now that

$$\lambda_1 = \cdots = \lambda_{k_1} > \lambda_{k_1+1} = \cdots = \lambda_{k_2} > \cdots > \lambda_{k_{s-1}+1} = \cdots = \lambda_{k_s} \quad (k_s = d).$$

From the fact proved above, the orthogonal projection onto $\text{span}\{u_1(p), \dots, u_{k_r}(p)\}$ converges for any $r = 1, \dots, s-1$, and this is trivial for $r = s$. Therefore, the orthogonal projection onto $\text{span}\{u_{k_{r-1}+1}(p), \dots, u_{k_r}(p)\}$ converges to a projection P_r for any $r = 1, \dots, s$, and thus Z_p converges to $\sum_{r=1}^s \lambda_{k_r} P_r$. \square

For $1 \leq k \leq d$ define η_k by the right-hand side of (2.8). Then Lemma 2.3 (see also the proof of Lemma 2.1) implies that, for $k = 1, \dots, d$,

$$\lambda_k = \frac{\eta_k}{\eta_{k-1}} \quad \text{if } \eta_k > 0$$

(where $\eta_0 := 1$), and $\lambda_k = 0$ if $\eta_k = 0$. So one can effectively compute the eigenvalues of $Z := \lim_{p \rightarrow \infty} Z_p$; however, it does not seem that there is a simple algebraic method to compute the limit matrix Z .

3 The maximal case

Let A and B be $d \times d$ positive semidefinite matrices with diagonalizations (2.1) and (2.2). For each $d \times d$ matrix X we write $s_1(X) \geq s_2(X) \geq \cdots \geq s_d(X)$ for the singular values of X in decreasing order with multiplicities. For each $p > 0$ and $k = 1, \dots, d$, since $\prod_{i=1}^k \lambda_i(p) = (\prod_{i=1}^k s_i(A^{p/2} B^{p/2}))^{2/p}$, by the majorization results of Gel'fand and Naimark and of Horn (see, e.g., [15, 5, 9]), we have

$$\prod_{j=1}^k a_{i_j} b_{n+1-i_j} \leq \prod_{j=1}^k \lambda_j(p) \leq \prod_{j=1}^k a_j b_j$$

for any choice of $1 \leq i_1 < i_2 < \cdots < i_k \leq d$, and for $k = d$

$$\prod_{i=1}^d \lambda_i(p) = \det A \cdot \det B = \prod_{i=1}^d a_i b_i.$$

That is, for any $p > 0$,

$$(a_i b_{n+1-i})_{i=1}^d \prec_{(\log)} (\lambda_i(p))_{i=1}^d \prec_{(\log)} (a_i b_i)_{i=1}^d \quad (3.1)$$

with the notation of log-majorization, see [2]. Letting $p \rightarrow \infty$ gives

$$(a_i b_{n+1-i})_{i=1}^d \prec_{(\log)} (\lambda_i)_{i=1}^d \prec_{(\log)} (a_i b_i)_{i=1}^d \quad (3.2)$$

for the eigenvalues $\lambda_1 \geq \dots \geq \lambda_d$ of $Z = \lim_{p \rightarrow \infty} Z_p$. In general, we have nothing to say about the position of $(\lambda_i)_{i=1}^d$ in (3.2). For instance, when V^*W becomes the permutation matrix corresponding to a permutation (j_1, \dots, j_d) of $(1, \dots, d)$, we have $Z_p = V \text{diag}(a_1 b_{j_1}, \dots, a_d b_{j_d}) V^*$ independently of $p > 0$ so that $(\lambda_i) = (a_i b_{j_i})$.

In this section we clarify the case when $(\lambda_i)_{i=1}^d$ is equal to $(a_i b_i)_{i=1}^d$, the maximal case in the log-majorization order in (3.2). To do this, let $0 = i_0 < i_1 < \dots < i_{l-1} < i_l = d$ and $0 = j_0 < j_1 < \dots < j_{m-1} < j_m = d$ be taken so that

$$\begin{aligned} a_1 &= \dots = a_{i_1} > a_{i_1+1} = \dots = a_{i_2} > \dots > a_{i_{l-1}+1} = \dots = a_{i_l}, \\ b_1 &= \dots = b_{j_1} > b_{j_1+1} = \dots = b_{j_2} > \dots > b_{j_{m-1}+1} = \dots = b_{j_m}. \end{aligned}$$

Theorem 3.1. *In the above situation the following conditions are equivalent:*

- (i) $\lambda_i = a_i b_i$ for all $i = 1, \dots, d$;
- (ii) for every $k = 1, \dots, d$ so that $i_{r-1} < k \leq i_r$ and $j_{s-1} < k \leq j_s$, there are $I_k, J_k \in \mathcal{I}_d(k)$ such that

$$\{1, \dots, i_{r-1}\} \subset I_k \subset \{1, \dots, i_r\}, \quad \{1, \dots, j_{s-1}\} \subset J_k \subset \{1, \dots, j_s\},$$

$$\det(V^*W)_{I_k, J_k} \neq 0;$$

- (iii) the property in (ii) holds for every $k \in \{i_1, \dots, i_{l-1}, j_1, \dots, j_{m-1}\}$.

Proof. (i) \Leftrightarrow (ii). By Lemma 2.3 condition (ii) means that

$$\prod_{i=1}^k \lambda_i = \prod_{i=1}^k a_i b_i, \quad k = 1, \dots, d.$$

It follows (see the proof of Lemma 2.1) that this is equivalent to (i).

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). By Lemma 2.3 again condition (iii) means that

$$\prod_{i=1}^h \lambda_i = \prod_{i=1}^h a_i b_i \quad \text{for all } h \in \{i_1, \dots, i_{l-1}, j_1, \dots, j_{m-1}\}. \quad (3.3)$$

This holds also for $h = d$ thanks to (3.2). We need to prove that $\prod_{i=1}^k \lambda_i = \prod_{i=1}^k a_i b_i$ for all $k = 1, \dots, d$. Now, let $i_{r-1} < k \leq i_r$ and $j_{s-1} < k \leq j_s$ as in condition (ii). If $k = i_r$ or $k = j_s$, then the conclusion has already been stated in (3.3). So assume that

$i_{r-1} < k < i_r$ and $j_{s-1} < k < j_s$. Set $h_0 := \max\{i_{r-1}, j_{s-1}\}$ and $h_1 := \min\{i_r, j_s\}$ so that $h_0 < k < h_1$. By (3.3) for $h = h_0, h_1$ we have

$$\prod_{i=1}^{h_0} \lambda_i = \prod_{i=1}^{h_0} a_i b_i > 0, \quad \prod_{i=1}^{h_1} \lambda_i = \prod_{i=1}^{h_1} a_i b_i.$$

Since $a_i = a_{h_1}$ and $b_i = b_{h_1}$ for $h_0 < i \leq h_1$, we have $\prod_{i=h_0+1}^{h_1} \lambda_i = (a_{h_1} b_{h_1})^{h_1-h_0}$. By (3.2) we furthermore have $\prod_{i=1}^{h_0+1} \lambda_i \leq \prod_{i=1}^{h_0+1} a_i b_i$ and hence

$$a_{h_1} b_{h_1} \geq \lambda_{h_0+1} \geq \lambda_{h_0+2} \geq \cdots \geq \lambda_{h_1}.$$

Therefore, $\lambda_i = a_{h_1} b_{h_1}$ for all i with $h_0 + 1 < i \leq h_1$, from which $\prod_{i=1}^k \lambda_i = \prod_{i=1}^k a_i b_i$ follows for $h_0 < k < h_1$. \square

Proposition 3.2. *Assume that the equivalent conditions of Theorem 3.1 hold. Then, for each $r = 1, \dots, l$, the spectral projection of Z corresponding to the set of eigenvalues $\{a_{i_{r-1}+1} b_{i_{r-1}+1}, \dots, a_{i_r} b_{i_r}\}$ is equal to the spectral projection $\sum_{i=i_{r-1}+1}^{i_r} v_i v_i^*$ of A corresponding to a_{i_r} . Hence Z is of the form*

$$Z = \sum_{i=1}^d a_i b_i u_i u_i^*$$

for some orthonormal set $\{u_1, \dots, u_d\}$ such that $\sum_{i=i_{r-1}+1}^{i_r} u_i u_i^* = \sum_{i=i_{r-1}+1}^{i_r} v_i v_i^*$ for $r = 1, \dots, l$.

Proof. In addition to Theorem 2.5 we may prove that, for each $k \in \{i_1, \dots, i_{l-1}\}$, the spectral projection of Z_p corresponding to $\{\lambda_1(p), \dots, \lambda_k(p)\}$ converges to $\sum_{i=1}^k v_i v_i^*$. Assume that $k = i_r$ with $1 \leq r \leq l-1$. When $j_{s-1} < k < j_s$, by condition (iii) of Theorem 3.1 we have $\det(V^*W)_{\{1, \dots, k\}, \{1, \dots, j_{s-1}, j'_s, \dots, j'_k\}} \neq 0$ for some $\{j'_s, \dots, j'_k\} \subset \{j_{s-1} + 1, \dots, j_s\}$. By exchanging $w_{j'_s}, \dots, w_{j'_k}$ with $w_{j_{s-1}+1}, \dots, w_k$ we may assume that $\det(V^*W)_{\{1, \dots, k\}, \{1, \dots, k\}} \neq 0$. Furthermore, by replacing A and B with $VA V^*$ and $VB V^*$, respectively, we may assume that $V = I$. So we end up assuming that

$$A = \text{diag}(a_1, \dots, a_d), \quad B = W \text{diag}(b_1, \dots, b_d) W^*,$$

and $\det W(1, \dots, k) \neq 0$, where $W(1, \dots, k)$ denotes the principal $k \times k$ submatrix of the top-left corner. Let $\{e_1, \dots, e_d\}$ be the standard basis of \mathbb{C}^d . By Theorem 3.1 we have

$$\lim_{p \rightarrow \infty} \lambda_1(Z_p^{\wedge k}) = \prod_{i=1}^k a_i b_i > \prod_{i=1}^{k-1} a_i b_i \cdot a_{k+1} b_{k+1} = \lim_{p \rightarrow \infty} \lambda_2(Z_p^{\wedge k})$$

so that the largest eigenvalue of $Z_p^{\wedge k}$ is simple for every sufficiently large p . Let $\{u_1(p), \dots, u_d(p)\}$ be an orthonormal basis of \mathbb{C}^d for which $Z_p u_i(p) = \lambda_i(p) u_i(p)$ for $1 \leq i \leq d$. Then $u_1(p) \wedge \cdots \wedge u_k(p)$ is the unit eigenvector of $Z_p^{\wedge k}$ corresponding to the

eigenvalue $\lambda_1(Z_p^{\wedge k})$. We now show that $u_1(p) \wedge \cdots \wedge u_k(p)$ converges to $e_1 \wedge \cdots \wedge e_k$ in $(\mathbb{C}^d)^{\wedge k}$. We observe that

$$(A^{\wedge k})^{p/2} = \text{diag}(a_I^{p/2})_I = a_{\{1, \dots, k\}}^{p/2} \text{diag}\left(1, \alpha_2^{p/2}, \dots, \alpha_{\binom{d}{k}}^{p/2}\right)$$

with respect to the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : I = \{i_1, \dots, i_k\} \in \mathcal{I}_d(k)\}$, where the first diagonal entry 1 corresponds to $e_1 \wedge \cdots \wedge e_k$ and $0 \leq \alpha_h < 1$ for $2 \leq h \leq \binom{d}{k}$. Similarly,

$$((\text{diag}(b_1, \dots, b_d))^{\wedge k})^p = b_{\{1, \dots, k\}}^p \text{diag}\left(1, \beta_2^p, \dots, \beta_{\binom{d}{k}}^p\right),$$

where $0 \leq \beta_h \leq 1$ for $2 \leq h \leq \binom{d}{k}$. Moreover, $W^{\wedge k}$ is given as

$$W^{\wedge k} = [w_{I,J}]_{I,J} = \begin{bmatrix} w_{11} & \cdots & w_{1\binom{d}{k}} \\ \vdots & \ddots & \vdots \\ w_{\binom{d}{k}1} & \cdots & w_{\binom{d}{k}\binom{d}{k}} \end{bmatrix},$$

where $w_{I,J} = \det W_{I,J}$ and so $w_{11} = \det W(1, \dots, k) \neq 0$. As in the proof of Theorem 2.5 we now compute

$$\begin{aligned} (Z_p^{\wedge k})^p &= (A^{\wedge k})^{p/2} W^{\wedge k} ((\text{diag}(b_1, \dots, b_d))^{\wedge k})^p (W^{\wedge k})^* (A^{\wedge k})^{p/2} \\ &= (a_{\{1, \dots, k\}} b_{\{1, \dots, k\}})^p \left[\sum_{h=1}^{\binom{d}{k}} w_{ih} \bar{w}_{jh} \alpha_i^{p/2} \alpha_j^{p/2} \beta_h^p \right]_{i,j=1}^{\binom{d}{k}}, \end{aligned}$$

where $\alpha_1 = \beta_1 = 1$. As $p \rightarrow \infty$ we have

$$\left[\sum_{h=1}^{\binom{d}{k}} w_{ih} \bar{w}_{jh} \alpha_i^{p/2} \alpha_j^{p/2} \beta_h^p \right] \rightarrow \text{diag}\left(\sum_{h:\beta_h=1} |w_{1h}|^2, 0, \dots, 0\right)$$

Since the unit eigenvector of $Z_p^{\wedge k}$ corresponding to the largest eigenvalue coincides with that of $\left[\sum_{h=1}^{\binom{d}{k}} w_{ih} \bar{w}_{jh} \alpha_i^{p/2} \alpha_j^{p/2} \beta_h^p \right]$, it follows that $u_1(p) \wedge \cdots \wedge u_k(p)$ converges to $e_1 \wedge \cdots \wedge e_k$ up to a scalar multiple $e^{\sqrt{-1}\theta}$, $\theta \in \mathbb{R}$. By Lemma 2.4 this implies the desired assertion. \square

Corollary 3.3. *If the eigenvalues a_1, \dots, a_d of A are all distinct and the conditions of Theorem 3.1 hold, then*

$$\lim_{p \rightarrow \infty} (A^{p/2} B^p A^{p/2})^{1/p} = V \text{diag}(a_1 b_1, a_2 b_2, \dots, a_d b_d) V^*.$$

In particular, when the eigenvalues of A are all distinct and so are those of B , the conditions of Theorem 3.1 means that all the leading principal minors of $V^* W$ are non-zero.

4 Extension to more than two matrices

Let A_1, \dots, A_m be $d \times d$ positive semidefinite matrices with diagonalizations

$$A_l = V_l D_l V_l^*, \quad D_l = \text{diag}(a_1^{(l)}, \dots, a_d^{(l)}), \quad 1 \leq l \leq m.$$

For each $p > 0$ consider the positive semidefinite matrix

$$\begin{aligned} Z_p &:= (A_1^{p/2} A_2^{p/2} \cdots A_{m-1}^{p/2} A_m^p A_{m-1}^{p/2} \cdots A_1^{p/2} A_1^{p/2})^{1/p}, \\ &= V_1 (D_1^{p/2} W_1 \cdots D_{m-1}^{p/2} W_{m-1} D_m^p W_{m-1}^* D_{m-1}^{p/2} \cdots W_1^* D_1^{p/2})^{1/p} V_1^*, \end{aligned}$$

where

$$W_l := V_l^* V_{l+1} = \left[w_{ij}^{(l)} \right]_{i,j=1}^d, \quad 1 \leq l \leq m-1.$$

The eigenvalues of Z_p are denoted as $\lambda_1(p) \geq \lambda_2(p) \geq \cdots \geq \lambda_d(p)$ in decreasing order. Although the log-majorization in (2.5) is no longer available in the present situation, we can extend Lemma 2.2 as follows.

Lemma 4.1. *The limit $\lambda_1 := \lim_{p \rightarrow \infty} \lambda_1(p)$ exists and*

$$\lambda_1 = \max \{ a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_m}^{(m)} : \mathbf{w}(i_1, i_2, \dots, i_m) \neq 0 \}, \quad (4.1)$$

where

$$\begin{aligned} &\mathbf{w}(i_1, i_2, \dots, i_m) \\ &:= \sum \left\{ w_{i_1 j_2}^{(1)} w_{j_2 j_3}^{(2)} \cdots w_{j_{m-1} i_m}^{(m)} : 1 \leq j_2, \dots, j_{m-1} \leq d, a_{j_2}^{(2)} \cdots a_{j_{m-1}}^{(m-1)} = a_{i_2}^{(2)} \cdots a_{i_{m-1}}^{(m-1)} \right\}. \end{aligned}$$

Moreover, $a_1^{(1)} \cdots a_1^{(m)} \geq \lambda_1 \geq a_d^{(1)} \cdots a_d^{(m)}$.

Proof. We notice that

$$\begin{aligned} [V_1^* Z_p^p V_1]_{ii} &= [D_1^{p/2} W_1 \cdots D_{m-1}^{p/2} W_{m-1} D_m^p W_{m-1}^* D_{m-1}^{p/2} \cdots W_1^* D_1^{p/2}]_{ii} \\ &= \sum_{i_2, \dots, i_{m-1}, k, j_{m-1}, \dots, j_2} (a_i^{(1)})^{p/2} w_{ii_2}^{(1)} (a_{i_2}^{(2)})^{p/2} \cdots w_{i_{m-2} i_{m-1}}^{(m-2)} (a_{i_{m-1}}^{(m-1)})^{p/2} \\ &\quad \times w_{i_{m-1} k}^{(m-1)} (a_k^{(m)})^p \overline{w_{j_{m-1} k}^{(m-1)}} (a_{j_{m-1}}^{(m-1)})^{p/2} \overline{w_{j_{m-2} j_{m-1}}^{(m-2)}} \cdots (a_{j_2}^{(2)})^{p/2} \overline{w_{ij_2}^{(1)}} (a_i^{(1)})^{p/2} \\ &= \sum_k \sum_{i_2, \dots, i_{m-1}} w_{ii_2}^{(1)} w_{i_2 i_3}^{(2)} \cdots w_{i_{m-1} k}^{(m-1)} (a_i^{(1)} a_{i_2}^{(2)} \cdots a_{i_{m-1}}^{(m-1)} a_k^{(m)})^{p/2} \\ &\quad \times \sum_{j_2, \dots, j_{m-1}} \overline{w_{ij_2}^{(1)} w_{j_2 j_3}^{(2)} \cdots w_{j_{m-1} k}^{(m-1)}} (a_i^{(1)} a_{j_2}^{(2)} \cdots a_{j_{m-1}}^{(m-1)} a_k^{(m)})^{p/2} \\ &= \sum_k \left| \sum_{j_2, \dots, j_{m-1}} w_{ij_2}^{(1)} w_{j_2 j_3}^{(2)} \cdots w_{j_{m-1} k}^{(m-1)} (a_i^{(1)} a_{j_2}^{(2)} \cdots a_{j_{m-1}}^{(m-1)} a_k^{(m)})^{p/2} \right|^2. \end{aligned}$$

Let η be the right-hand side of (4.1). From the above expression we have

$$\begin{aligned}\lambda_1(p)^p &\leq \text{Tr } V_1^* Z_p^p V_1 \\ &= \sum_{i,k} \left| \sum_{j_2, \dots, j_{m-1}} w_{ij_2}^{(1)} w_{j_2 j_3}^{(2)} \cdots w_{j_{m-1} k}^{(m-1)} (a_i^{(1)} a_{j_2}^{(2)} \cdots a_{j_{m-1}}^{(m-1)} a_k^{(m)})^{p/2} \right|^2 \\ &\leq M \eta^p,\end{aligned}$$

where $M > 0$ is a constant independent of p . Therefore, $\limsup_{p \rightarrow \infty} \lambda_1(p) \leq \eta$. On the other hand, let $(i, i_2, \dots, i_{m-1}, k)$ be such that $a_i^{(1)} a_{i_2}^{(2)} \cdots a_{i_{m-1}}^{(m-1)} a_k^{(m)} = \eta$, and let $\delta := |\mathbf{w}(i, i_2, \dots, i_{m-1}, k)| > 0$. Then we have

$$\left| \sum_{j_2, \dots, j_{m-1}} w_{ij_2}^{(1)} w_{j_2 j_3}^{(2)} \cdots w_{j_{m-1} k}^{(m-1)} (a_i^{(1)} a_{j_2}^{(2)} \cdots a_{j_{m-1}}^{(m-1)} a_k^{(m)})^{p/2} \right| \geq \delta \eta^{p/2} - M' \alpha^{p/2}$$

for some constants $M' > 0$ and $\alpha > 0$ with $\alpha < \eta$. Therefore, for sufficiently large p we have $\delta \eta^{p/2} - M' \alpha^{p/2} > 0$ and

$$d \lambda_1(p)^p \geq \text{Tr } V_1^* Z_p^p V_1 \geq (\delta \eta^{p/2} - M' \alpha^{p/2})^2 = \delta^2 \eta^p \left(1 - \frac{M'}{\delta} \left(\frac{\alpha}{\eta} \right)^{p/2} \right)^2$$

so that $\liminf_{p \rightarrow \infty} \lambda_1(p) \geq \eta$. The latter assertion is obvious. \square

Lemma 4.2. *For every $i = 1, \dots, d$ the limit $\lambda_i := \lim_{p \rightarrow \infty} \lambda_i(p)$ exists.*

Proof. For every $k = 1, \dots, d$ apply Lemma 4.1 to $A_1^{\wedge k}, \dots, A_m^{\wedge k}$ to see that

$$\lim_{p \rightarrow \infty} \lambda_1(p) \lambda_2(p) \cdots \lambda_k(p)$$

exists. Hence, the limit $\lim_{p \rightarrow \infty} \lambda_i(p)$ exists for $i = 1, \dots, d$ as in the proof of Lemma 2.1. \square

Theorem 4.3. *For every $d \times d$ positive semidefinite matrices A_1, \dots, A_m the matrix*

$$Z_p = (A_1^{p/2} A_2^{p/2} \cdots A_{m-1}^{p/2} A_m^p A_{m-1}^{p/2} \cdots A_2^{p/2} A_1^{p/2})^{1/p}$$

converges as $p \rightarrow \infty$.

Proof. The proof is similar to that of Theorem 2.5. Choose an orthogonal basis $\{u_1(p), \dots, u_d(p)\}$ of \mathbb{C}^d such that $Z_p u_i(p) = \lambda_i(p) u_i(p)$ for $1 \leq i \leq d$. Let k ($1 \leq k < d$) be such that $\lambda_1 \geq \cdots \geq \lambda_k > \lambda_{k+1}$. Since (2.9) holds in the present case too, $\lambda_1(Z_p^{\wedge k})$ is a simple eigenvalue of $Z_p^{\wedge k}$ for every p sufficiently large. For $I, J \in \mathcal{I}_d(k)$ we write $w_{I,J}^{(l)} := \det W_{I,J}^{(l)}$ for $1 \leq l \leq m-1$ and $a_I^{(l)} := \prod_{i \in I} a_i^{(l)}$ for $1 \leq l \leq m$. We have

$$[V_1^{*\wedge k} (Z_p^{\wedge k})^p V_1^{\wedge k}]_{I,J}$$

$$\begin{aligned}
&= \sum_{K \in \mathcal{I}_d(k)} \sum_{I_2, \dots, I_{m-1}} w_{I, I_2}^{(1)} w_{I_2, I_3}^{(2)} \cdots w_{I_{m-1}, K}^{(m-1)} (a_I^{(1)} a_{I_2}^{(2)} \cdots a_{I_{m-1}}^{(m-1)} a_K^{(m)})^{p/2} \\
&\quad \times \sum_{J_2, \dots, J_{m-1}} \overline{w_{J, J_2}^{(1)} w_{J_2, J_3}^{(2)} \cdots w_{J_{m-1}, K}^{(m-1)}} (a_J^{(1)} a_{J_2}^{(2)} \cdots a_{J_{m-1}}^{(m-1)} a_K^{(m)})^{p/2} \\
&= \eta_k^p \sum_{K \in \mathcal{I}_d(k)} \sum_{I_2, \dots, I_{m-1}} w_{I, I_2}^{(1)} w_{I_2, I_3}^{(2)} \cdots w_{I_{m-1}, K}^{(m-1)} \left(\frac{a_I^{(1)} a_{I_2}^{(2)} \cdots a_{I_{m-1}}^{(m-1)} a_K^{(m)}}{\eta_k} \right)^{p/2} \\
&\quad \times \sum_{J_2, \dots, J_{m-1}} \overline{w_{J, J_2}^{(1)} w_{J_2, J_3}^{(2)} \cdots w_{J_{m-1}, K}^{(m-1)}} \left(\frac{a_J^{(1)} a_{J_2}^{(2)} \cdots a_{J_{m-1}}^{(m-1)} a_K^{(m)}}{\eta_k} \right)^{p/2},
\end{aligned}$$

where

$$\eta_k := \lambda_1 \lambda_2 \cdots \lambda_k = \max \{ a_{I_1}^{(1)} a_{I_2}^{(2)} \cdots a_{I_{m-1}}^{(m-1)} a_{I_m}^{(m)} : \mathbf{w}_k(I_1, I_2, \dots, I_{m-1}, I_m) \neq 0 \}$$

and

$$\begin{aligned}
&\mathbf{w}_k(I_1, I_2, \dots, I_{m-1}, I_m) \\
&:= \sum \left\{ w_{I_1, I_2}^{(1)} w_{I_2, I_3}^{(2)} \cdots w_{I_{m-1}, I_m}^{(m-1)} : J_2, \dots, J_{m-1} \in \mathcal{I}_d(k), a_{J_2}^{(2)} \cdots a_{J_{m-1}}^{(m-1)} = a_{I_2}^{(2)} \cdots a_{I_{m-1}}^{(m-1)} \right\}.
\end{aligned}$$

We see that

$$V_1^{*\wedge k} \left(\frac{Z_p^{\wedge k}}{\eta_k} \right)^p V_1^{\wedge k} \longrightarrow Q := \left[\sum_{K \in \mathcal{I}_d(k)} \mathbf{v}_k(I, K) \overline{\mathbf{v}_k(J, K)} \right]_{I, J} \quad \text{as } p \rightarrow \infty,$$

where

$$\mathbf{v}_k(I, K) := \mathbf{w}_k(I, I_2, \dots, I_{m-1}, K)$$

if $\mathbf{w}_k(I, I_2, \dots, I_{m-1}, K) \neq 0$ and $a_I^{(1)} a_{I_2}^{(2)} \cdots a_{I_{m-1}}^{(m-1)} a_K^{(m)} = \eta_k$ for some $I_2, \dots, I_{m-1} \in \mathcal{I}_d(k)$, and otherwise $\mathbf{v}_k(I, K) := 0$. Since $Q_{I, I} \geq |\mathbf{v}_k(I, K)|^2 > 0$ for some $I, K \in \mathcal{I}_d(k)$, note that $Q \neq 0$. The remaining proof is the same as in that of Theorem 2.5. \square

5 Limit of $(A^p \# B^p)^{1/p}$ as $p \rightarrow \infty$

Another problem, seemingly more interesting, is to know what is shown on the convergence $(A^p \sigma B^p)^{1/p}$ as $p \rightarrow \infty$, the anti-version of (1.2) (or Theorem B.1). For example, when $\sigma = \nabla$, the arithmetic mean, the increasing limit of $(A^p \nabla B^p)^{1/p} = ((A^p + B^p)/2)^{1/p}$ as $p \rightarrow \infty$ exists and

$$A \vee B := \lim_{p \rightarrow \infty} (A^{-p} \nabla B^{-p})^{-1/p} = \lim_{p \rightarrow \infty} (A^p + B^p)^{1/p} \quad (5.1)$$

is the supremum of A, B with respect to some spectral order among Hermitian matrices, see [12] and [1, Lemma 6.5]. When $\sigma = !$, the harmonic mean, we have the infimum counterpart $A \wedge B := \lim_{p \rightarrow \infty} (A^p ! B^p)^{1/p}$, the decreasing limit.

In this section we are interested in the case where $\sigma = \#$, the geometric mean. For each $p > 0$ and $d \times d$ positive semidefinite matrices A, B with the diagonalizations in (2.1) and (2.2) we define

$$G_p := (A^p \# B^p)^{2/p}, \quad (5.2)$$

which is given as $(A^{p/2}(A^{-p/2}B^pA^{-p/2})^{1/2}A^{p/2})^{2/p}$ if $A > 0$. The eigenvalues of G_p are denoted as $\lambda_1(G_p) \geq \dots \geq \lambda_d(G_p)$ in decreasing order.

Proposition 5.1. *For every $i = 1, \dots, d$ the limit*

$$\widehat{\lambda}_i := \lim_{p \rightarrow \infty} \lambda_i(G_p)$$

exists, and $a_1b_1 \geq \widehat{\lambda}_1 \geq \dots \geq \widehat{\lambda}_d \geq a_db_d$. Furthermore,

$$(a_ib_{d+1-i})_{i=1}^d \prec_{(\log)} (\widehat{\lambda}_i)_{i=1}^d \prec_{(\log)} (a_ib_i)_{i=1}^d. \quad (5.3)$$

Proof. Since $(a_1b_1)^{p/2}I \geq A^p \# B^p \geq (a_db_d)^{p/2}I$, we have $a_1b_1 \geq \lambda_i(G_p) \geq a_db_d$ for all $i = 1, \dots, d$ and $p > 0$. By the log-majorization result in [2, Theorem 2.1], for every $k = 1, \dots, d$ we have

$$\prod_{i=1}^k \lambda_i(G_p) \geq \prod_{i=1}^k \lambda_i(G_q) \quad \text{if } 0 < p < q. \quad (5.4)$$

This implies that the limit of $\prod_{i=1}^k \lambda_i(G_p)$ as $p \rightarrow \infty$ exists for every $k = 1, \dots, d$, and hence the limit $\lambda_i(G_p)$ exists for $i = 1, \dots, d$ as in the proof of Lemma 2.1.

To prove the latter assertion, it suffices to show that

$$(a_ib_{d+1-i})_{i=1}^d \prec_{(\log)} (\lambda_i(G_1))_{i=1}^d \prec_{(\log)} (a_ib_i)_{i=1}^d \quad (5.5)$$

for $G_1 = (A \# B)^2$. Indeed, applying this to A^p and B^p we have

$$(a_ib_{d+1-i})_{i=1}^d \prec_{(\log)} (\lambda_i(G_p))_{i=1}^d \prec_{(\log)} (a_ib_i)_{i=1}^d$$

so that (5.3) follows by letting $p \rightarrow \infty$. To prove (5.5), we may by continuity assume that $A > 0$. By [2, Corollary 2.3] and (3.1) we have

$$(\lambda_i(G_1))_{i=1}^d \prec_{(\log)} (\lambda_i(A^{1/2}BA^{1/2}))_{i=1}^d \prec_{(\log)} (a_ib_i)_{i=1}^d.$$

Since $G_1^{1/2}A^{-1}G_1^{1/2} = B$, there exists a unitary matrix V such that $A^{-1/2}G_1A^{-1/2} = VBV^*$ and hence $G_1 = A^{1/2}VBV^*A^{1/2}$. Since $\lambda_i(VBV^*) = b_i$, by the majorization of Gel'fand and Naimark we have

$$(a_ib_{d+1-i})_{i=1}^d \prec_{(\log)} (\lambda_i(G_1))_{i=1}^d,$$

proving (5.5) □

In view of (2.5) and (5.4) we may consider G_p as the complementary counterpart of Z_p in some sense; yet it is also worth noting that G_p is symmetric in A and B while Z_p is not. Our ultimate goal is to prove the existence of the limit of G_p in (5.2) as $p \rightarrow \infty$ similarly to Theorem 2.5 and to clarify, similarly to Theorem 3.1, the minimal case when $(\widehat{\lambda}_i)_{i=1}^d$ is equal to the decreasing rearrangement of $(a_i b_{d+1-i})_{i=1}^d$. However, the problem seems much more difficult, and we can currently settle the special case of 2×2 matrices only.

Proposition 5.2. *Let A and B be 2×2 positive semidefinite matrices with the diagonalizations (2.1) and (2.2) with $d = 2$. Then G_p in (5.2) converges as $p \rightarrow \infty$ to a positive semidefinite matrix whose eigenvalues are*

$$(\widehat{\lambda}_1, \widehat{\lambda}_2) = \begin{cases} (a_1 b_1, a_2 b_2) & \text{if } (V^* W)_{12} = 0, \\ (\max\{a_1 b_2, a_2 b_1\}, \min\{a_1 b_2, a_2 b_1\}) & \text{if } (V^* W)_{12} \neq 0. \end{cases}$$

Proof. Since

$$G_p = V((\text{diag}(a_1, a_2))^p \# (V^* W \text{diag}(b_1, b_2) V^* W)^p)^{2/p} V^*,$$

we may assume without loss of generality that $V = I$ (then $V^* W = W$).

First, when $W_{12} = 0$ (hence W is diagonal), we have for every $p > 0$

$$G_p = \text{diag}(a_1 b_1, a_2 b_2).$$

Next, when $W_{11} = 0$ (hence $W = \begin{bmatrix} 0 & w_1 \\ w_2 & 0 \end{bmatrix}$ with $|w_1| = |w_2| = 1$), we have for every $p > 0$

$$G_p = \text{diag}(a_1 b_2, a_2 b_1).$$

In the rest it suffices to consider the case where $W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ with $w_{ij} \neq 0$ for all $i, j = 1, 2$. First, assume that $\det A = \det B = 1$ so that $a_1 a_2 = b_1 b_2 = 1$. For every $p > 0$, since $\det A^p = \det B^p = 1$, it is known [16, Proposition 3.11] (also [6, Proposition 4.1.12]) that

$$A^p \# B^p = \frac{A^p + B^p}{\sqrt{\det(A^p + B^p)}}$$

so that

$$G_p = \frac{(A^p + B^p)^{2/p}}{(\det(A^p + B^p))^{1/p}}.$$

Compute

$$A^p + B^p = \begin{bmatrix} a_1^p + |w_{11}|^2 b_1^p + |w_{12}|^2 b_2^p & w_{11} \overline{w_{21}} b_1^p + w_{12} \overline{w_{22}} b_2^p \\ \overline{w_{11}} w_{21} b_1^p + \overline{w_{12}} w_{22} b_2^p & a_2^p + |w_{21}|^2 b_1^p + |w_{22}|^2 b_2^p \end{bmatrix} \quad (5.6)$$

and

$$\det(A^p + B^p) = 1 + |w_{21}|^2 (a_1 b_1)^p + |w_{22}|^2 (a_1 b_2)^p + |w_{11}|^2 (a_2 b_1)^p + |w_{12}|^2 (a_2 b_2)^p$$

$$+ |w_{11}w_{22} - w_{12}w_{21}|^2. \quad (5.7)$$

Hence we have

$$\lim_{p \rightarrow \infty} (\det(A^p + B^p))^{1/p} = a_1 b_1, \quad \lim_{p \rightarrow \infty} (\operatorname{Tr}(A^p + B^p))^{1/p} = \max\{a_1, b_1\}.$$

Therefore, thanks to (5.1) we have

$$\lim_{p \rightarrow \infty} G_p = \frac{(A \vee B)^2}{a_1 b_1}.$$

Since

$$\frac{1}{2} \operatorname{Tr}(A^p \# B^p) \leq (\lambda_1(G_p))^{p/2} \leq \operatorname{Tr}(A^p \# B^p),$$

we obtain

$$\begin{aligned} \widehat{\lambda}_1 &= \lim_{p \rightarrow \infty} (\operatorname{Tr}(A^p \# B^p))^{2/p} = \lim_{p \rightarrow \infty} \frac{(\operatorname{Tr}(A^p + B^p))^{2/p}}{(\det(A^p + B^p))^{1/p}} \\ &= \frac{\max\{a_1^2, b_1^2\}}{a_1 b_1} = \max\left\{\frac{a_1}{b_1}, \frac{b_1}{a_1}\right\} = \max\{a_1 b_2, a_2 b_1\}. \end{aligned}$$

Furthermore, $\widehat{\lambda}_2 = \min\{a_1 b_2, a_2 b_1\}$ follows since $\widehat{\lambda}_1 \widehat{\lambda}_2 = 1$.

For general $A, B > 0$ let $\alpha := \sqrt{\det A}$ and $\beta := \sqrt{\det B}$. Since

$$G_p = \alpha \beta ((\alpha^{-1} A)^p \# (\beta^{-1} B)^p)^{2/p},$$

we see from the above case that G_p converges as $p \rightarrow \infty$ and

$$\widehat{\lambda}_1 = \alpha \beta \max\{(\alpha^{-1} a_1)(\beta^{-1} b_2), (\alpha^{-1} a_2)(\beta^{-1} b_1)\} = \max\{a_1 b_2, a_2 b_1\},$$

and similarly for $\widehat{\lambda}_2$.

The remaining is the case when a_2 and/or $b_2 = 0$. We may assume that $a_1, b_1 > 0$ since the case $A = 0$ or $B = 0$ is trivial. When $a_2 = b_2 = 0$, since $a_1^{-1}A$ and $b_1^{-1}B$ are non-commuting rank one projections, we have $G_p = 0$ for all $p > 0$ by [13, (3.11)]. Finally, assume that $a_2 = 0$ and $B > 0$. Then we may assume that $a_1 = 1$ and $\det B = 1$. For $\varepsilon > 0$ set $A_\varepsilon := \operatorname{diag}(1, \varepsilon^2)$. Since $\det(\varepsilon^{-1}A_\varepsilon) = 1$, we have

$$A_\varepsilon^p \# B^p = \varepsilon^{p/2} ((\varepsilon^{-1}A_\varepsilon)^p \# B^p) = \varepsilon^{p/2} \frac{(\varepsilon^{-1}A_\varepsilon)^p + B^p}{\sqrt{\det((\varepsilon^{-1}A_\varepsilon)^p + B^p)}}.$$

By use of (5.6) and (5.7) with $a_1 = \varepsilon^{-1}$ and $a_2 = \varepsilon$ we compute

$$A^p \# B^p = \lim_{\varepsilon \searrow 0} A_\varepsilon^p \# B^p = (|w_{21}|^2 b_1^p + |w_{22}|^2 b_2^p)^{-1/2} \operatorname{diag}(1, 0)$$

so that

$$\lim_{p \rightarrow \infty} G_p = \operatorname{diag}(b_1^{-1}, 0) = \operatorname{diag}(b_2, 0),$$

which is the desired assertion in this final situation. \square

A Proof of Lemma 2.4

We may assume that $\mathcal{H} = \mathbb{C}^d$ by fixing an orthonormal basis of \mathcal{H} . Let $G(k, d)$ denote the Grassmannian manifold consisting of k -dimensional subspaces of \mathcal{H} . Let $\mathcal{O}_{k,d}$ denote the set of all $u = (u_1, \dots, u_k) \in \mathcal{H}^k$ such that u_1, \dots, u_k are orthonormal in \mathcal{H} . Consider $\mathcal{O}_{k,d}$ as a metric space with the metric

$$d_2(u, v) := \left(\sum_{i=1}^k \|u_i - v_i\|^2 \right)^{1/2}, \quad u = (u_1, \dots, u_k), \quad v = (v_1, \dots, v_k) \in \mathcal{H}^k.$$

Moreover, let $\tilde{\mathcal{H}}_{k,d}$ be the set of projectivised vectors $u = u_1 \wedge \dots \wedge u_k$ in $\mathcal{H}^{\wedge k}$ of norm 1, i.e., the quotient space of $\mathcal{H}_{k,d} := \{u \in \mathcal{H}^{\wedge k} : u = u_1 \wedge \dots \wedge u_k, \|u\| = 1\}$ under the equivalent relation $u \sim v$ on $\mathcal{H}_{k,d}$ defined as $u = e^{i\theta}v$ for some $\theta \in \mathbb{R}$. We then have the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{k,d} & \xrightarrow{\pi} & G(k, d) \\ & \searrow \tilde{\pi} & \downarrow \phi \\ & & \tilde{\mathcal{H}}_{k,d} \end{array}$$

where π and $\tilde{\pi}$ are surjective maps defined for $u = (u_1, \dots, u_k) \in \mathcal{O}_{k,d}$ as

$$\begin{aligned} \pi(u) &:= \text{span}\{u_1, \dots, u_k\}, \\ \tilde{\pi}(u) &:= [u_1 \wedge \dots \wedge u_k], \text{ the equivalence class of } u_1 \wedge \dots \wedge u_k, \end{aligned}$$

and ϕ is the canonical representation of $G(k, d)$ by the k th antisymmetric tensors (or the k th exterior products).

As shown in [7], the standard Grassmannian topology on $G(k, d)$ is the final topology (the quotient topology) from the map π and it coincides with the topology induced by the gap metric:

$$d_{\text{gap}}(\mathcal{U}, \mathcal{V}) := \|P_{\mathcal{U}} - P_{\mathcal{V}}\|$$

for k -dimensional subspaces \mathcal{U}, \mathcal{V} of \mathcal{H} and the orthogonal projections $P_{\mathcal{U}}, P_{\mathcal{V}}$ onto them. On the other hand, consider the quotient topology on $\tilde{\mathcal{H}}_{k,d}$ induced from the norm on $\mathcal{H}_{k,d} \subset \mathcal{H}^{\wedge k}$, which is determined by the metric

$$\tilde{d}(\tilde{\pi}(u), \tilde{\pi}(v)) := \inf_{\theta \in \mathbb{R}} \|u_1 \wedge \dots \wedge u_k - e^{\sqrt{-1}\theta} v_1 \wedge \dots \wedge v_k\|, \quad u, v \in \mathcal{O}_{k,d}.$$

It is easy to prove that $\tilde{\pi} : (\mathcal{O}_{k,d}, d_2) \rightarrow (\tilde{\mathcal{H}}_{k,d}, \tilde{d})$ is continuous. Since $(\mathcal{O}_{k,d}, d_2)$ is compact, it thus follows that the final topology on $\tilde{\mathcal{H}}_{k,d}$ from the map $\tilde{\pi}$ coincides with the \tilde{d} -topology.

It is clear from the above commutative diagram that the final topology on $G(k, d)$ from π is homeomorphic via ϕ to that on $\tilde{\mathcal{H}}_{k,d}$ from $\tilde{\pi}$. Hence ϕ is a homeomorphism

from $(G(k, d), d_{\text{gap}})$ onto $(\tilde{\mathcal{H}}_{k,d}, \tilde{d})$. From the homogeneity of $(G(k, d), d_{\text{gap}})$ and $(\tilde{\mathcal{H}}_{k,d}, \tilde{d})$ under the unitary transformations there exist constant $\alpha, \beta > 0$ (depending on only k, d) such that

$$\alpha \|P_{\pi(u)} - P_{\pi(v)}\| \leq \tilde{d}(\tilde{\pi}(u), \tilde{\pi}(v)) \leq \beta \|P_{\pi(u)} - P_{\pi(v)}\|, \quad u, v \in \mathcal{O}_{k,d},$$

which is the desired inequality.

B Proof of (1.2)

This appendix is aimed to supply the proof of (1.2) for matrices $A, B \geq 0$. Throughout the appendix let A, B be $d \times d$ positive semidefinite matrices with the support projections A^0, B^0 . We define $\log A$ in the generalized sense as

$$\log A := (\log A)A^0,$$

i.e., $\log A$ is defined by the usual functional calculus on the range of A^0 and it is zero on the range of $A^{0\perp} = I - A^0$, and similarly $\log B := (\log B)B^0$. We write $P_0 := A^0 \wedge B^0$ and

$$\log A \dot{+} \log B := P_0(\log A)P_0 + P_0(\log B)P_0.$$

Note [10, Section 4] that

$$\begin{aligned} P_0 \exp(\log A \dot{+} \log B) &= \lim_{\varepsilon \searrow 0} \exp(\log(A + \varepsilon A^{0\perp}) + \log(B + \varepsilon B^{0\perp})) \\ &= \lim_{\varepsilon \searrow 0} \exp(\log(A + \varepsilon I) + \log(B + \varepsilon I)). \end{aligned} \quad (\text{B.1})$$

Now, let σ be an operator mean with the representing operator monotone function f on $(0, \infty)$, and let $\alpha := f'(1)$. Note that $0 \leq \alpha \leq 1$ and if $\alpha = 0$ (resp., $\alpha = 1$) then $A \sigma B = A$ (resp., $A \sigma B = B$) so that $(A^p \sigma B^p)^{1/p} = A$ (resp., $(A^p \sigma B^p)^{1/p} = B$) for all $A, B \geq 0$ and $p > 0$. So in the rest we assume that $0 < \alpha < 1$.

Theorem B.1. *With the above assumptions, for every $A, B \geq 0$,*

$$\lim_{p \searrow 0} (A^p \sigma B^p)^{1/p} = P_0 \exp((1 - \alpha) \log A \dot{+} \alpha \log B). \quad (\text{B.2})$$

From (B.1) we may write

$$\begin{aligned} \lim_{p \searrow 0} (A^p \sigma B^p)^{1/p} &= \lim_{\varepsilon \searrow 0} \exp((1 - \alpha) \log(A + \varepsilon I) + \alpha \log(B + \varepsilon I)) \\ &= \lim_{\varepsilon \searrow 0} \lim_{p \searrow 0} ((A + \varepsilon I)^p \sigma (B + \varepsilon I)^p)^{1/p}. \end{aligned}$$

The next lemma is essential to prove the theorem. The proof of the lemma is a slight modification of that of [10, Lemma 4.1].

Lemma B.2. For each $p \in (0, p_0)$ with some $p_0 > 0$, a Hermitian matrix $Z(p)$ is given in the 2×2 block form as

$$Z(p) = \begin{bmatrix} Z_0(p) & Z_2(p) \\ Z_2^*(p) & Z_1(p) \end{bmatrix},$$

where $Z_0(p)$ is $k \times k$, $Z_1(p)$ is $l \times l$ and $Z_2(p)$ is $k \times l$. Assume:

- (a) $Z_0(p) \rightarrow Z_0$ and $Z_2(p) \rightarrow Z_2$ as $p \searrow 0$,
- (b) there is a $\delta > 0$ such that $pZ_1(p) \leq -\delta I_l$ for all $p \in (0, p_0)$.

Then

$$e^{Z(p)} \longrightarrow \begin{bmatrix} e^{Z_0} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{as } p \searrow 0.$$

Proof. We list the eigenvalues of $Z(p)$ in decreasing order (with multiplicities) as

$$\lambda_1(p) \geq \cdots \geq \lambda_k(p) \geq \lambda_{k+1}(p) \geq \cdots \geq \lambda_m(p)$$

together with the corresponding orthonormal eigenvectors

$$u_1(p), \dots, u_k(p), u_{k+1}(p), \dots, u_m(p),$$

where $m := k + l$. Then

$$e^{Z(p)} = \sum_{i=1}^m e^{\lambda_i(p)} u_i(p) u_i(p)^*. \quad (\text{B.3})$$

Furthermore, let $\mu_1(p) \geq \cdots \geq \mu_k(p)$ be the eigenvalues of $Z_0(p)$ and $\mu_1 \geq \cdots \geq \mu_k$ be the eigenvalues of Z_0 . Then $\mu_i(p) \rightarrow \mu_i$ as $p \searrow 0$ thanks to assumption (a). By the majorization result for eigenvalues in [1, Corollary 7.2] we have

$$\sum_{i=1}^r \mu_i(p) \leq \sum_{i=1}^r \lambda_i(p), \quad 1 \leq r \leq k. \quad (\text{B.4})$$

Since

$$pZ(p) \leq \begin{bmatrix} pZ_0(p) & pZ_2(p) \\ pZ_2^*(p) & -\delta I_l \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & -\delta I_l \end{bmatrix} \quad \text{as } p \searrow 0$$

thanks to assumptions (a) and (b), it follows that, for $k < i \leq m$, $p\lambda_i(p) < -\delta/2$ for any $p > 0$ sufficiently small so that

$$\lim_{p \searrow 0} \lambda_i(p) = -\infty, \quad k < i \leq m. \quad (\text{B.5})$$

Hence, it suffices to prove that for any sequence $(p_0 >) p_n \searrow 0$ there exists a subsequence $\{p'_n\}$ of $\{p_n\}$ such that we have for $1 \leq i \leq k$

$$\lambda_i(p'_n) \longrightarrow \mu_i \quad \text{as } n \rightarrow \infty, \quad (\text{B.6})$$

$$u_i(p'_n) \longrightarrow v_i \oplus 0 \in \mathbb{C}^k \oplus \mathbb{C}^l \quad \text{as } n \rightarrow \infty, \quad (\text{B.7})$$

$$Z_0 v_i = \mu_i v_i. \quad (\text{B.8})$$

Indeed, it then follows that v_1, \dots, v_k are orthonormal vectors in \mathbb{C}^k , so from (B.3) and (B.5) we obtain

$$\lim_{n \rightarrow \infty} e^{Z(p'_n)} = \sum_{i=1}^k e^{\mu_i} v_i v_i^* \oplus 0 = e^{Z_0} \oplus 0.$$

Now, replacing $\{p_n\}$ with a subsequence, we may assume that $u_i(p_n)$ itself converges to some $u_i \in \mathbb{C}^m$ for $1 \leq i \leq k$. Writing $u_i(p_n) = v_i^{(n)} \oplus w_i^{(n)}$ in $\mathbb{C}^k \oplus \mathbb{C}^l$, we have

$$\begin{aligned} \lambda_1(p_n) &= \langle v_i^{(n)} \oplus w_i^{(n)}, Z(p_n)(v_i^{(n)} \oplus w_i^{(n)}) \rangle \\ &= \langle v_i^{(n)}, Z_0(p_n) v_i^{(n)} \rangle + 2\operatorname{Re} \langle v_i^{(n)}, Z_2(p_n) w_i^{(n)} \rangle + \langle w_i^{(n)}, Z_1(p_n) w_i^{(n)} \rangle \\ &\leq \langle v_i^{(n)}, Z_0(p_n) v_i^{(n)} \rangle + 2\operatorname{Re} \langle v_i^{(n)}, Z_2(p_n) w_i^{(n)} \rangle - \frac{\delta}{p_n} \|w_i^{(n)}\|^2 \end{aligned} \quad (\text{B.9})$$

due to assumption (b). For $i = 1$, since $\mu_1(p_n) \leq \lambda_1(p_n)$ by (B.4) for $r = 1$, it follows from (B.9) that

$$p_n \mu_1(p_n) \leq p_n \|Z_0(p_n)\| + 2p_n \|Z_2(p_n)\| - \delta \|w_1^{(n)}\|^2,$$

where $\|Z_0(p_n)\|$ and $\|Z_2(p_n)\|$ are the operator norms. As $n \rightarrow \infty$ ($p_n \searrow 0$), by assumption (a) we have $w_1^{(n)} \rightarrow 0$ so that $u_1(p_n) \rightarrow u_1 = v_1 \oplus 0$ in $\mathbb{C}^k \oplus \mathbb{C}^l$. From (B.9) again we furthermore have

$$\limsup_{n \rightarrow \infty} \lambda_1(p_n) \leq \langle v_1, Z_0 v_1 \rangle \leq \mu_1 \leq \liminf_{n \rightarrow \infty} \lambda_1(p_n)$$

since $\mu_1(p_n) \leq \lambda_1(p_n)$ and $\mu_1(p_n) \rightarrow \mu_1$. Therefore, $\lambda_1(p_n) \rightarrow \langle v_1, Z_0 v_1 \rangle = \mu_1$ and hence $Z_0 v_1 = \mu_1 v_1$. Next, when $k \geq 2$ and $i = 2$, since $\lambda_2(p_n)$ is bounded below by (B.4) for $r = 2$, it follows as above that $w_2^{(n)} \rightarrow 0$ and hence $u_2(p_n) \rightarrow u_2 = v_2 \oplus 0$. Therefore,

$$\limsup_{n \rightarrow \infty} \lambda_2(p_n) \leq \langle v_2, Z_0 v_2 \rangle \leq \mu_2 \leq \liminf_{n \rightarrow \infty} \lambda_2(p_n)$$

so that $\lambda_2(p_n) \rightarrow \langle v_2, Z_0 v_2 \rangle = \mu_2$ and $Z_0 v_2 = \mu_2 v_2$, since μ_2 is the largest eigenvalue of Z_0 restricted to $\{v_1\}^\perp \cap \mathbb{C}^k$. Repeating this argument we obtain (B.6)–(B.8) for $1 \leq i \leq k$. \square

Note that the lemma and its proof hold true even when the assumption $Z_2(p) \rightarrow Z_2$ in (b) is slightly relaxed into $p^{1/3} Z_2(p) \rightarrow 0$ as $p \searrow 0$. (For this, from (B.9) note that $p_n^{-1/3} w_i^{(n)} \rightarrow 0$ and so $Z_2(p_n) w_i^{(n)} \rightarrow 0$.)

Proof of Theorem B.1. Let us divide the proof into two steps. In the proof below we denote by ∇_α and $!_\alpha$ the weighted arithmetic and harmonic operator means having the representing functions $(1 - \alpha) + \alpha x$ and $x / ((1 - \alpha)x + \alpha)$, respectively. Note that

$$A !_\alpha B \leq A \sigma B \leq A \nabla_\alpha B, \quad A, B \geq 0.$$

Step 1. First, we prove the theorem in the case where $P \sigma Q = P \wedge Q$ for all orthogonal projections P, Q (this is the case, for instance, when σ is the weighted harmonic operator mean $!_\alpha$, see [13, Theorem 3.7]). Let \mathcal{H}_0 be the range of P_0 ($= A^0 !_\alpha B^0 = A^0 \sigma B^0$). From the operator monotonicity of $\log x$ ($x > 0$) it follows that, for every $p > 0$,

$$\frac{1}{p} \log(A^p !_\alpha B^p)|_{\mathcal{H}_0} \leq \frac{1}{p} \log(A^p \sigma B^p)|_{\mathcal{H}_0} \leq \frac{1}{p} \log(P_0(A^p \nabla_\alpha B^p)P_0)|_{\mathcal{H}_0}. \quad (\text{B.10})$$

For every $\varepsilon > 0$ we have

$$\begin{aligned} (A + \varepsilon A^{0\perp})^p !_\alpha (B + \varepsilon B^{0\perp})^p &= ((A + \varepsilon A^{0\perp})^{-p} \nabla_\alpha (B + \varepsilon B^{0\perp})^{-p})^{-1} \\ &= (A^{-p} \nabla_\alpha B^{-p} + \varepsilon^{-p} (A^{0\perp} \nabla_\alpha B^{0\perp}))^{-1}, \end{aligned}$$

where $A^{-p} = (A^{-1})^p$ and $B^{-p} = (B^{-1})^p$ are taken as the generalized inverses. Therefore,

$$\begin{aligned} P_0((A + \varepsilon A^{0\perp})^p !_\alpha (B + \varepsilon B^{0\perp})^p)P_0 &\geq (P_0(A^{-p} \nabla_\alpha B^{-p} + \varepsilon^{-p} (A^{0\perp} \nabla_\alpha B^{0\perp}))P_0)^{-1} \\ &= (P_0(A^{-p} \nabla_\alpha B^{-p})P_0)^{-1}, \end{aligned} \quad (\text{B.11})$$

since the support projection of $A^{0\perp} + B^{0\perp}$ is $A^{0\perp} \vee B^{0\perp} = P_0^\perp$. In the above, $(-)^{-1}$ is the generalized inverse (with support \mathcal{H}_0) and the inequality follows from the operator convexity of x^{-1} ($x > 0$). Letting $\varepsilon \searrow 0$ in (B.11) gives

$$A^p !_\alpha B^p = P_0(A^p !_\alpha B^p)P_0 \geq (P_0(A^{-p} \nabla_\alpha B^{-p})P_0)^{-1}$$

so that

$$\frac{1}{p} \log(A^p !_\alpha B^p)|_{\mathcal{H}_0} \geq -\frac{1}{p} \log(P_0(A^{-p} \nabla_\alpha B^{-p})P_0)|_{\mathcal{H}_0}. \quad (\text{B.12})$$

Combining (B.10) and (B.12) yields

$$-\frac{1}{p} \log(P_0(A^{-p} \nabla_\alpha B^{-p})P_0)|_{\mathcal{H}_0} \leq \frac{1}{p} \log(A^p \sigma B^p)|_{\mathcal{H}_0} \leq \frac{1}{p} \log(P_0(A^p \nabla_\alpha B^p)P_0)|_{\mathcal{H}_0}. \quad (\text{B.13})$$

Since

$$A^{-p} = A^0 - p \log A + o(p), \quad B^{-p} = B^0 - p \log B + o(p)$$

as $p \searrow 0$, we have

$$A^{-p} \nabla_\alpha B^{-p} = A^0 \nabla_\alpha B^0 - p((\log A) \nabla_\alpha (\log B)) + o(p)$$

so that

$$P_0(A^{-p} \nabla_\alpha B^{-p})P_0 = P_0 - p((1 - \alpha) \log A + \alpha \log B) + o(p).$$

Therefore,

$$-\frac{1}{p} \log(P_0(A^{-p} \nabla_\alpha B^{-p})P_0)|_{\mathcal{H}_0} = ((1 - \alpha) \log A + \alpha \log B)|_{\mathcal{H}_0} + o(1). \quad (\text{B.14})$$

Similarly,

$$\frac{1}{p} \log(P_0(A^p \nabla B^p)P_0)|_{\mathcal{H}_0} = ((1 - \alpha) \log A + \alpha \log B)|_{\mathcal{H}_0} + o(1). \quad (\text{B.15})$$

From (B.13)–(B.15) we obtain

$$\lim_{p \searrow 0} \frac{1}{p} \log(A^p \sigma B^p)|_{\mathcal{H}_0} = ((1 - \alpha) \log A + \alpha \log B)|_{\mathcal{H}_0},$$

which yields the required limit formula.

Step 2. For a general operator mean σ the integral representation theorem [13, Theorem 4.4] says that there are $0 \leq \theta \leq 1$, $0 \leq \beta \leq 1$ and an operator mean τ such that

$$\sigma = \theta \nabla_\beta + (1 - \theta) \tau$$

and $P \tau Q = P \wedge Q$ for all orthogonal projections P, Q . Moreover, τ has the representing operator monotone function g on $(0, \infty)$ for which $\gamma := g'(1) \in (0, 1)$ and

$$\alpha = \theta \beta + (1 - \theta) \gamma.$$

We may assume that $0 < \theta \leq 1$ since the case $\theta = 0$ was shown in Step 1. Moreover, when $\theta = 1$, we have $\beta = \alpha \in (0, 1)$. For the present, assume that $0 < \theta \leq 1$ and $0 < \beta < 1$. Let $A, B \geq 0$ be given, and note that $A^0 \sigma B^0 = \theta A^0 \nabla_\beta B^0 + (1 - \theta)(A^0 \wedge B^0)$ has the support projection $A^0 \vee B^0$. Let \mathcal{H} , \mathcal{H}_0 and \mathcal{H}_1 denote the ranges of $A^0 \vee B^0$, $P_0 = A^0 \wedge B^0$ and $A^0 \vee B^0 - P_0$, respectively, so that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Note that the support of $A^p \sigma B^p$ for any $p > 0$ is \mathcal{H} . We will describe $\frac{1}{p} \log(A^p \sigma B^p)|_{\mathcal{H}}$ in the 2×2 block form with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Let

$$Z_0 := ((1 - \gamma) \log A + \gamma \log B)|_{\mathcal{H}_0}.$$

It follows from Step 1 that $\lim_{p \searrow 0} (A^p \tau B^p)^{1/p} = P_0 e^{Z_0} P_0$ and hence

$$\begin{aligned} A^p \tau B^p &= P_0 (e^{Z_0} + o(1))^p P_0 \\ &= P_0 (I_{\mathcal{H}_0} + p \log(e^{Z_0} + o(1)) + o(p)) P_0 \\ &= P_0 (I_{\mathcal{H}_0} + p Z_0 + o(p)) P_0 \\ &= P_0 + p((1 - \gamma) \log A + \gamma \log B) + o(p). \end{aligned}$$

In the above, the third equality follows since $\log(e^{Z_0} + o(1)) = Z_0 + o(1)$. On the other hand, we have

$$A^p \nabla_\beta B^p = A^0 \nabla_\beta B^0 + p((\log A) \nabla_\beta (\log B)) + o(p).$$

Therefore, we have

$$A^p \sigma B^p = \theta(A^0 \nabla_\beta B^0) + (1 - \theta)P_0$$

$$+ p\theta((\log A) \nabla_\beta(\log B)) + p(1 - \theta)((1 - \gamma) \log A + \gamma \log B) + o(p).$$

Setting

$$\begin{aligned} C &:= (\theta(A^0 \nabla_\beta B^0) + (1 - \theta)P_0)|_{\mathcal{H}}, \\ H &:= (\theta((\log A) \nabla_\beta(\log B)) + (1 - \theta)((1 - \gamma) \log A + \gamma \log B))|_{\mathcal{H}}, \end{aligned}$$

we write

$$\frac{1}{p} \log(A^p \sigma B^p)|_{\mathcal{H}} = \frac{1}{p} \log(C + pH + o(p)), \quad (\text{B.16})$$

which C is a positive definite contraction on \mathcal{H} and H is a Hermitian operator on \mathcal{H} . Note that the eigenspace of C for the eigenvalue 1 is \mathcal{H}_0 . Hence, with a basis consisting of orthonormal eigenvectors for C we may assume that C is diagonal so that $C = \text{diag}(c_1, \dots, c_m)$ with

$$c_1 = \dots = c_k = 1 > c_{k+1} \geq \dots \geq c_m > 0$$

where $m = \dim \mathcal{H}$ and $k = \dim \mathcal{H}_0$.

Applying the Taylor formula (see, e.g., [9, Theorem 2.3.1] to $\log(C + pH + o(p))$ we have

$$\log(C + pH + o(p)) = \log C + pD \log(C)(H) + o(p), \quad (\text{B.17})$$

where $D \log(C)$ denotes the Fréchet derivative of the matrix functional calculus by $\log x$ at C . The Daleckii and Krein's derivative formula (see, e.g., [9, Theorem 2.3.1]) says that

$$D \log(C)(H) = \left[\frac{\log c_i - \log c_j}{c_i - c_j} \right]_{i,j=1}^m \circ H, \quad (\text{B.18})$$

where \circ denotes the Schur (or Hadamard) product and $(\log c_i - \log c_j)/(c_i - c_j)$ is understood as $1/c_i$ when $c_i = c_j$. We write $D \log(C)(H)$ in the 2×2 block form on $\mathcal{H}_0 \oplus \mathcal{H}_1$ as $\begin{bmatrix} Z_0 & Z_2 \\ Z_2^* & Z_1 \end{bmatrix}$ where $Z_0 := P_0 H P_0|_{\mathcal{H}_0}$. By (B.16)–(B.18) we can write

$$\frac{1}{p} \log(A^p \sigma B^p) = \frac{1}{p} \log C + D \log(C)(H) + o(1) = \begin{bmatrix} Z_0(p) & Z_2(p) \\ Z_2^*(p) & Z_1(p) \end{bmatrix},$$

where

$$\begin{aligned} Z_0(p) &= Z_0 + o(1), & Z_2(p) &= Z_2 + o(1), \\ Z_1(p) &= \frac{1}{p} \text{diag}(\log c_{k+1}, \dots, \log c_m) + Z_1 + o(1). \end{aligned}$$

This 2×2 block form of $Z(p) := \frac{1}{p} \log(A^p \sigma B^p)|_{\mathcal{H}}$ satisfies assumptions (a) and (b) of Lemma B.2 for $p \in (0, p_0)$ with a sufficiently small $p_0 > 0$. Therefore, the lemma implies that

$$\lim_{p \searrow 0} (A^p \sigma B^p)^{2/p}|_{\mathcal{H}} = \lim_{p \searrow 0} \exp \left(\frac{1}{p} \log(A^p \sigma B^p)|_{\mathcal{H}} \right) = e^{Z_0} \oplus 0$$

on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Since

$$\begin{aligned} Z_0 &= P_0 H P_0|_{\mathcal{H}_0} = \theta((1 - \beta) \log A + \beta \log B) + (1 - \theta)((1 - \gamma) \log A + \gamma \log B) \\ &= (1 - \alpha) \log A + \alpha \log B, \end{aligned}$$

we obtain the desired limit formula.

For the remaining case where $0 < \theta < 1$ and $\beta = 0$ or 1 the proof is similar to the above when we take as \mathcal{H} the range of A^0 (for $\beta = 0$) or B^0 (for $\beta = 1$) instead of the range of $A^0 \vee B^0$. \square

Finally, we remark that the same method as in the proof of Step 2 above can also be applied to give an independent proof of (1.1) for matrices $A, B \geq 0$.

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